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General Necessary Conditions for the Derivation of the Secondary Constraints of a First-Order Relativistic Wave Equation

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A first-order relativistic wave equation of the Gel'fand-Yaglom form is considered in the presence of an external electromagnetic field, and general conditions are derived that are necessary for the derivation of its secondary constraints, the general form of which is given. Examples demonstrating the validity of these conditions are also given.

1. INTRODUCTION

Relativistic wave equations of the first order with the general form (Gel'fand *et al.*, 1963)

$$\{-\mathbb{L}_0\pi_0+\mathbb{L}_1\pi_1+\mathbb{L}_2\pi_2+\mathbb{L}_3\pi_3+\mathbb{1}\chi\}\psi=0 \tag{1}$$

accept subsidiary conditions of the second kind (secondary constraints) (Pauli and Fierz, 1939; Velo and Zwanziger, 1969; Koutroulos, 1986a–c) namely, relations among the components of the wave function ψ involving no derivatives. In the above equation, $\mathbb{L}_i$ $(i=0, 1, 2, 3)$ are $n\times n$ matrices, their dimension n depending on the underlying representation according to which the wave function ψ transforms; $\mathbb{1}$ is the $n\times n$ unit matrix; χ is a constant related to the masses of the particles associated with the field described by the wave equation; and π_i $(i=0, 1, 2, 3)$ are the components of the electromagnetic four-momentum vector. We consider the above wave equation in the presence of an external electromagnetic field.

A major problem with these wave equations is how the subsidiary conditions associated with them can be derived. In certain cases it is preferable to reformulate the wave equation in spinorial form and then, by

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employing spinor calculus, to obtain its subsidiary conditions, as was done in Koutroulos (1986b,c) (see Pauli and Fierz, 1939). The method of using spinor calculus is very complicated. Thus, it is better to look for other ways of finding the subsidiary conditions, such as by means of matrix calculus, as is demonstrated in Koutroulos (1986a).

In the present paper we consider the general first-order wave equation (1) and derive general conditions that have to be satisfied in order that the wave equation possess subsidiary conditions of the second kind.

2. GENERAL CONDITIONS

In this section we see what conditions have to be satisfied in order to be able to derive the subsidiary conditions of the second kind for a first-order relativistic wave equation. Let us consider the first-order relativistic wave equation (1) realized in an n-dimsnsional space and let us multiply it from the left by the expression

$$\mathbb{S}(\mathbb{A}_0\pi_0+\mathbb{A}_1\pi_1+\mathbb{A}_2\pi_2+\mathbb{A}_3\pi_3+\mathbb{A}_4\chi) \tag{2}$$

to obtain

$$\begin{aligned}(\mathbb{S}\mathbb{A}_0\pi_0+\mathbb{S}\mathbb{A}_1\pi_1+\mathbb{S}\mathbb{A}_2\pi_2+\mathbb{S}\mathbb{A}_3\pi_3+\mathbb{S}\mathbb{A}_4\chi)\\ \times(-\mathbb{L}\pi_0+\mathbb{L}_1\pi_1+\mathbb{L}_2\pi_2+\mathbb{L}_3\pi_3+\mathbb{1}\chi)\psi=0\end{aligned} \tag{3}$$

In the above expressions $\mathbb{S}$ is a $1\times n$ matrix and $\mathbb{A}_l$, $l=0, 1, 2, 3, 4$, are $n\times n$ matrices. Noticing that the products $\mathbb{S}\mathbb{A}_p$, $p=0, 1, 2, 3, 4$, are $1\times n$ matrices and that they can be identified with the row vectors $\boldsymbol{\alpha}_p$, $p=0, 1, 2, 3, 4$, we can rewrite (3) as follows:

$$\begin{aligned}\boldsymbol{\alpha}_0\pi_0+\boldsymbol{\alpha}_1\pi_1+\boldsymbol{\alpha}_2\pi_2+\boldsymbol{\alpha}_3\pi_3+\boldsymbol{\alpha}_4\chi)\\ \times(\mathbb{L}_0\pi+\mathbb{L}_1\pi_1+\mathbb{L}_2\pi_2+\mathbb{L}_3\pi_3+\mathbb{1}\chi)\psi=0\end{aligned} \tag{4}$$

or as follows:

$$\begin{aligned}&-\boldsymbol{\alpha}_0\mathbb{L}_0\pi_0^2\psi+\boldsymbol{\alpha}_0\mathbb{L}_1\pi_0\pi_1\psi+\boldsymbol{\alpha}_0\mathbb{L}_2\pi_0\pi_2\psi+\boldsymbol{\alpha}_0\mathbb{L}_3\pi_0\pi_3\psi\\ &\quad+\boldsymbol{\alpha}_0\pi_0\mathbb{1}\chi\psi-\boldsymbol{\alpha}_1\mathbb{L}_0\pi_1\pi_0\psi+\boldsymbol{\alpha}_1\mathbb{L}_1\pi_1^2\psi+\boldsymbol{\alpha}_1\mathbb{L}_2\pi_1\pi_2\psi+\boldsymbol{\alpha}_1\mathbb{L}_3\pi_1\pi_3\psi\\ &\quad+\boldsymbol{\alpha}_1\pi_1\mathbb{1}\chi\psi-\boldsymbol{\alpha}_2\mathbb{L}_0\pi_2\pi_0\psi+\boldsymbol{\alpha}_2\mathbb{L}_1\pi_2\pi_1\psi+\boldsymbol{\alpha}_2\mathbb{L}_2\pi_2^2\psi\\ &\quad+\boldsymbol{\alpha}_2\mathbb{L}_3\pi_2\pi_3\psi+\boldsymbol{\alpha}_2\pi_2\mathbb{1}\chi\psi-\boldsymbol{\alpha}_3\mathbb{L}_0\pi_3\pi_0\psi+\boldsymbol{\alpha}_3\mathbb{L}_1\pi_3\pi_1\psi\\ &\quad+\boldsymbol{\alpha}_3\mathbb{L}_2\pi_3\pi_2\psi+\boldsymbol{\alpha}_3\mathbb{L}_3\pi_3^2\psi+\boldsymbol{\alpha}_3\pi_3\mathbb{1}\chi\psi-\boldsymbol{\alpha}_4\chi\mathbb{L}_0\pi_0\psi\\ &\quad+\boldsymbol{\alpha}_4\chi\mathbb{L}_1\pi_1\psi+\boldsymbol{\alpha}_4\chi\mathbb{L}_2\pi_2\psi+\boldsymbol{\alpha}_4\chi\mathbb{L}_3\pi_3\psi+\boldsymbol{\alpha}_4\chi\mathbb{1}\chi\psi=0\end{aligned} \tag{5}$$

This expression can become a subsidiary condition of the second kind if certain conditions are satisfied. To find these conditions, let us consider first the terms in (5) linear in π_i, $i=0, 1, 2, 3$, namely

$$\begin{aligned} &(-\boldsymbol{\alpha}_4\mathbb{L}_0+\boldsymbol{\alpha}_0)\pi_0\chi\psi, \qquad (\boldsymbol{\alpha}_4\mathbb{L}_1+\boldsymbol{\alpha}_1)\pi_1\chi\psi \\ &(\boldsymbol{\alpha}_4\mathbb{L}_2+\boldsymbol{\alpha}_2)\pi_2\chi\psi, \qquad (\boldsymbol{\alpha}_4\mathbb{L}_3+\boldsymbol{\alpha}_3)\pi_3\chi\psi \end{aligned} \tag{6}$$

Terms like these must not appear in the subsidiary conditions of the second kind, and we eliminate them by requiring that

$$\begin{aligned} &\boldsymbol{\alpha}_4\mathbb{L}_0=\boldsymbol{\alpha}_0, \qquad \boldsymbol{\alpha}_4\mathbb{L}_1=-\boldsymbol{\alpha}_1 \\ &\boldsymbol{\alpha}_4\mathbb{L}_2=-\boldsymbol{\alpha}_2, \quad \boldsymbol{\alpha}_4\mathbb{L}_3=-\boldsymbol{\alpha}_3 \end{aligned} \tag{7}$$

Then let us consider the terms in (5) involving π_i^2, $i=0, 1, 2, 3$, namely

$$\boldsymbol{\alpha}_0\mathbb{L}_0\pi_0^2\psi, \quad \boldsymbol{\alpha}_1\mathbb{L}_1\pi_1^2\psi, \quad \boldsymbol{\alpha}_2\mathbb{L}_2\pi_2^2\psi, \quad \boldsymbol{\alpha}_3\mathbb{L}_3\pi_3^2\psi \tag{8}$$

Such terms must disappear as well, and we obtain this by requiring that

$$\boldsymbol{\alpha}_0\mathbb{L}_0=0, \qquad \boldsymbol{\alpha}_1\mathbb{L}_1=0, \qquad \boldsymbol{\alpha}_2\mathbb{L}_2=0, \qquad \boldsymbol{\alpha}_3\mathbb{L}_3=0 \tag{9}$$

Finally, the terms in (5) involving $\pi_i\pi_j$, $i=0, 1, 2, 3$, $j=0, 1, 2, 3$, can be written, using the commutation relations

$$[\pi_k, \pi_l]=ieF_{kl}\equiv f_{kl}, \qquad k=0,1,2,3, \quad l=0,1,2,3 \tag{10}$$

(where F_{kl} is the electromagnetic field tensor, $i=\sqrt{-1}$), as follows:

$$\begin{aligned} \boldsymbol{\alpha}_0\mathbb{L}_1\pi_0\pi_1-\boldsymbol{\alpha}_1\mathbb{L}_0\pi_1\pi_0\psi &= (\boldsymbol{\alpha}_0\mathbb{L}_1-\boldsymbol{\alpha}_1\mathbb{L}_0)\pi_0\pi_1\psi-\boldsymbol{\alpha}_1\mathbb{L}_0f_{10}\psi \\ \boldsymbol{\alpha}_0\mathbb{L}_2\pi_0\pi_2\psi-\boldsymbol{\alpha}_2\mathbb{L}_0\pi_2\pi_0\psi &= (\boldsymbol{\alpha}_0\mathbb{L}_2-\boldsymbol{\alpha}_2\mathbb{L}_0)\pi_0\pi_2\psi-\boldsymbol{\alpha}_2\mathbb{L}_0f_{20}\psi \\ \boldsymbol{\alpha}_0\mathbb{L}_3\pi_0\pi_3\psi-\boldsymbol{\alpha}_3\mathbb{L}_0\pi_3\pi_0\psi &= (\boldsymbol{\alpha}_0\mathbb{L}_3-\boldsymbol{\alpha}_3\mathbb{L}_0)\pi_0\pi_3\psi-\boldsymbol{\alpha}_3\mathbb{L}_0f_{30}\psi \\ \boldsymbol{\alpha}_1\mathbb{L}_2\pi_1\pi_2\psi+\boldsymbol{\alpha}_2\mathbb{L}_1\pi_2\pi_1\psi &= (\boldsymbol{\alpha}_1\mathbb{L}_2+\boldsymbol{\alpha}_2\mathbb{L}_1)\pi_1\pi_2\psi+\boldsymbol{\alpha}_2\mathbb{L}_1f_{21}\psi \\ \boldsymbol{\alpha}_1\mathbb{L}_3\pi_1\pi_3\psi+\boldsymbol{\alpha}_3\mathbb{L}_1\pi_3\pi_1\psi &= (\boldsymbol{\alpha}_1\mathbb{L}_3+\boldsymbol{\alpha}_3\mathbb{L}_1)\pi_1\pi_3\psi+\boldsymbol{\alpha}_3\mathbb{L}_1f_{31}\psi \\ \boldsymbol{\alpha}_2\mathbb{L}_3\pi_2\pi_3\psi+\boldsymbol{\alpha}_3\mathbb{L}_2\pi_3\pi_2\psi &= (\boldsymbol{\alpha}_2\mathbb{L}_3+\boldsymbol{\alpha}_3\mathbb{L}_2)\pi_2\pi_3\psi+\boldsymbol{\alpha}_3\mathbb{L}_2f_{32}\psi \end{aligned} \tag{11}$$

The terms involving $\pi_i\pi_j$ on the right-hand side of the above relations should vanish and we obtain this by imposing the conditions

$$\begin{aligned} &\boldsymbol{\alpha}_0\mathbb{L}_1-\boldsymbol{\alpha}_1\mathbb{L}_0=0, \qquad \boldsymbol{\alpha}_0\mathbb{L}_2-\boldsymbol{\alpha}_2\mathbb{L}_0=0 \\ &\boldsymbol{\alpha}_0\mathbb{L}_3-\boldsymbol{\alpha}_3\mathbb{L}_0=0, \qquad \boldsymbol{\alpha}_1\mathbb{L}_2+\boldsymbol{\alpha}_2\mathbb{L}_1=0 \\ &\boldsymbol{\alpha}_1\mathbb{L}_3+\boldsymbol{\alpha}_3\mathbb{L}_1=0, \qquad \boldsymbol{\alpha}_2\mathbb{L}_3+\boldsymbol{\alpha}_3\mathbb{L}_0=0 \end{aligned} \tag{12}$$

Thus, expression (5), after imposing the conditions (7), (9), and (12), is reduced to the expression

$$(-\boldsymbol{\alpha}_1\mathbb{L}_0f_{10}-\boldsymbol{\alpha}_2\mathbb{L}_0f_{20}-\boldsymbol{\alpha}_3\mathbb{L}_0f_{30}+\boldsymbol{\alpha}_2\mathbb{L}_1f_{21}+\boldsymbol{\alpha}_3\mathbb{L}_1f_{31}+\boldsymbol{\alpha}_3\mathbb{L}_2f_{32}+\boldsymbol{\alpha}_4\chi^2\mathbb{1})\psi=0 \tag{13}$$

which is the general form of a subsidiary condition of the second kind for a first-order relativistic wave equation.

3. OTHER FORMS OF CONDITIONS (9) AND (12)

The above conditions (9) and (12) necessary for the derivation of the subsidiary conditions of the second kind can be expressed in other forms also. For instance, taking the transpose of (9), we have

$$\begin{aligned} \mathbb{L}_0^{tr}\boldsymbol{\alpha}_0^{tr} &= 0, \qquad \mathbb{L}_1^{tr}\boldsymbol{\alpha}_1^{tr} = 0 \\ \mathbb{L}_2^{tr}\boldsymbol{\alpha}_2^{tr} &= 0, \qquad \mathbb{L}_3^{tr}\boldsymbol{\alpha}_3^{tr} = 0 \end{aligned} \tag{14}$$

Written in this form, conditions (9) can be interpreted as follows: Each vector $\boldsymbol{\alpha}_i^{tr}$ for $i = 0, 1, 2, 3$ is an eigenvector of each matrix $\mathbb{L}_i^{tr}$ corresponding to zero eigenvalue.

We give applications of this statement below.

1. *Dirac wave equation.* Applying formulae (14) to the case of the Dirac wave equation, we see immediately that this equation does not accept subsidiary conditions of the second kind, because its matrices γ_i^{tr}, $i = 0, 1, 2, 3$, do not have zero eigenvalues.

2. *Bhabha wave equations.* For the same reason, the Bhabha wave equations for half-integer spin do not accept subsidiary conditions of the second kind.

Conditions (9) can also acquire another form. Thus, introducing (7) into (9) and considering the transpose, we have

$$\begin{aligned} (\mathbb{L}_0^{tr})^2\boldsymbol{\alpha}_4^{tr} &= 0 \qquad (\mathbb{L}_1^{tr})^2\boldsymbol{\alpha}_4^{tr} = 0 \\ (\mathbb{L}_2^{tr})^2\boldsymbol{\alpha}_4^{tr} &= 0 \qquad (\mathbb{L}_3^{tr})^2\boldsymbol{\alpha}_4^{tr} = 0 \end{aligned} \tag{15}$$

These relations can be understood as saying that the vector $\boldsymbol{\alpha}_4^{tr}$ is simultaneously an eigenvector of the matrices $(\mathbb{L}_i^{tr})^2$, $i = 0, 1, 2, 3$, corresponding to zero eigenvalue. Also introducing (7) into (12) and taking the transpose, we have the relations

$$\begin{aligned} [\mathbb{L}_1^{tr}, \mathbb{L}_0^{tr}]_+\boldsymbol{\alpha}_4^{tr} &= 0, \qquad [\mathbb{L}_2^{tr}, \mathbb{L}_0^{tr}]_+\boldsymbol{\alpha}_4^{tr} = 0 \\ [\mathbb{L}_3^{tr}, \mathbb{L}_0^{tr}]_+\boldsymbol{\alpha}_4^{tr} &= 0, \qquad [\mathbb{L}_2^{tr}, \mathbb{L}_1^{tr}]_+\boldsymbol{\alpha}_4^{tr} = 0 \\ [\mathbb{L}_3^{tr}, \mathbb{L}_1^{tr}]_+\boldsymbol{\alpha}_4^{tr} &= 0, \qquad [\mathbb{L}_3^{tr}, \mathbb{L}_2^{tr}]_+\boldsymbol{\alpha}_4^{tr} = 0 \end{aligned} \tag{16}$$

where the notation $[\ ,\ ]_+$ is used for the anticommutator between two matrices. These relations can be translated also as saying that the vector $\boldsymbol{\alpha}_4^{tr}$ is an eigenvector of the matrices

$$[\mathbb{L}_k^{tr}, \mathbb{L}_0^{tr}]_+, \qquad [\mathbb{L}_m^{tr}, \mathbb{L}_1^{tr}]_+, \qquad [\mathbb{L}_3^{tr}, \mathbb{L}_2^{tr}]_+, \qquad k = 1, 2, 3, \qquad m = 2, 3$$

corresponding to zero eigenvalue.

Conditions (16) can acquire yet another form if for this purpose we make use of the generators of rotations $\mathbb{A}_r$, $r=1, 2, 3$, and of the generators of boosts $\mathbb{B}_s$, $s=1, 2, 3$, and also of the relations

$$\mathbb{L}_2=-[\mathbb{L}_1, \mathbb{A}_3]_-, \qquad \mathbb{L}_3=[\mathbb{L}_1, \mathbb{A}_2]_-, \qquad \mathbb{L}_3=-[\mathbb{L}_2, \mathbb{A}_1]_-$$

$$\mathbb{L}_k=-[\mathbb{B}_k, \mathbb{L}_0]_-, \qquad k=1, 2, 3$$

where $[\,,\,]_-$ indicates the commutator between two matrices. The new form of conditions (16) is

$$\begin{aligned} &(\mathbb{L}_0^{\text{tr}})^2\mathbb{B}_1^{\text{tr}}\boldsymbol{\alpha}_4^{\text{tr}}=0, \qquad (\mathbb{L}_0^{\text{tr}})^2\mathbb{B}_2^{\text{tr}}\boldsymbol{\alpha}_4^{\text{tr}}=0 \\ &(\mathbb{L}_0^{\text{tr}})^2\mathbb{B}_3^{\text{tr}}\boldsymbol{\alpha}_4^{\text{tr}}=0, \qquad (\mathbb{L}_1^{\text{tr}})^2\mathbb{A}_3^{\text{tr}}\boldsymbol{\alpha}_4^{\text{tr}}=0 \\ &(\mathbb{L}_1^{\text{tr}})^2\mathbb{A}_2^{\text{tr}}\boldsymbol{\alpha}_4=0, \qquad (\mathbb{L}_2^{\text{tr}})^2\mathbb{A}_1^{\text{tr}}\boldsymbol{\alpha}_4^{\text{tr}}=0 \end{aligned} \tag{17}$$

These relations can be interpreted as saying that the vectors $\mathbb{B}_k^{\text{tr}}\boldsymbol{\alpha}^{\text{tr}}$, $k=1$, 2, 3, are eigenvectors of the matrix $(\mathbb{L}_0^{\text{tr}})^2$ corresponding to zero eigenvalue. The vectors $\mathbb{A}_m^{\text{tr}}\boldsymbol{\alpha}_4^{\text{tr}}$, $m=2, 3$, are eigenvectors of the matris $(\mathbb{L}_1^{\text{tr}})^2$ corresponding to zero eigenvalue; and finally the vector $\mathbb{A}_1^{\text{tr}}\boldsymbol{\alpha}_4^{\text{tr}}$ is an eigenvector of the matrix $(\mathbb{L}_2^{\text{tr}})^2$ corresponding to zero eigenvalue.

4. EXAMPLES

We now give examples of wave equations accepting subsidiary conditions of the second kind and satisfying all the previous conditions.

Example 1 (Pauli–Fierz wave equation). In the case of the Pauli–Fierz wave equation, conditions (7), (15) and (17) or (7), (14), and (16) are all satisfied, as can be seen by using the explicit form of the matrices $\mathbb{L}_i$, $i=0$, 1, 2, 3, and of the generators $\mathbb{A}_k$, $\mathbb{B}_k$, $k=1, 2, 3$, of the wave equation expressed in the spinor basis

$$\begin{aligned} \{&\alpha_{11}^{\dot{1}}, \alpha_{12}^{\dot{1}}, \alpha_{22}^{\dot{1}}, \alpha_{11}^{\dot{2}}, \alpha_{12}^{\dot{2}}, \alpha_{22}^{\dot{2}}, d^{\dot{1}}, d^{\dot{2}}, b_1^{\dot{1}\dot{1}}, b_1^{\dot{1}\dot{2}}, \\ &b_1^{\dot{2}\dot{2}}, b_2^{\dot{1}\dot{1}}, b_2^{\dot{1}\dot{2}}, b_2^{\dot{2}\dot{2}}, c_1, c_2\} \end{aligned} \tag{18}$$

which turns out to be more convenient to work with.

There are four subsidiary conditions of the second kind and their number is equal to the number of the spin-1/2 components $d^{\dot{1}}$, $d^{\dot{2}}$, c_1, c_2. These components must be constrained in order that the equation be a spin-3/2 wave equation.

The number of subsidiary conditions of the second kind according to the analysis in Sections 2 and 3 is also equal to the number of linearly independent eigenvectors $\boldsymbol{\alpha}_4^{\text{tr}}$ corresponding to the zero eigenvalues of the matrices $(\mathbb{L}_i^{\text{tr}})^2$, $i=0, 1, 2, 3$, and such that they also satisfy the conditions

(17), namely $\mathbb{B}_k^{\text{tr}}\boldsymbol{\alpha}_4^{\text{tr}}$, $k=1, 2, 3$, are eigenvectors of the matrix $(\mathbb{L}_0^{\text{tr}})^2$ corresponding to zero eigenvalue, $\mathbb{A}_m^{\text{tr}}\boldsymbol{\alpha}_4^{\text{tr}}$, $m=2, 3$, are eigenvectors of the matrix $(\mathbb{L}_1^{\text{tr}})^2$ corresponding to zero eigenvalue, and $\mathbb{A}_1^{\text{tr}}\boldsymbol{\alpha}_4^{\text{tr}}$ is an eigenvector of the matrix $(\mathbb{L}_2^{\text{tr}})^2$ corresponding to zero eigenvalue.

In the spinor basis (18) the matrices $(\mathbb{L}_i^{\text{tr}})^2$, $i=0, 1, 2, 3$, have zero rows and columns corresponding to the spin-$\frac{1}{2}$ components d^1, d^2, c_1, c_2, and so the eigenvectors $(\boldsymbol{\alpha}_4)_j^{\text{tr}}$, $j=1, 2, 3, 4$, of $(\mathbb{L}_i^{\text{tr}})^2$, $i=0, 1, 2, 3$, corresponding to zero eigenvalues and satisfying also (17) are

$$
(\boldsymbol{\alpha}_4^{\text{tr}})_1=\left[\begin{array}{c}0\\0\\0\\0\\0\\0\\ \hline 1\\0\\ \hline 0\\0\\0\\0\\0\\0\\ \hline 0\\0\end{array}\right]\begin{array}{l}\\\\\\\\\\\\ \text{spin-}\tfrac{1}{2}\\ \text{part}\\\\\\\\\\\\\\ \text{spin-}\tfrac{1}{2}\\ \text{part}\end{array},\quad
(\boldsymbol{\alpha}_4^{\text{tr}})_2=\left[\begin{array}{c}0\\0\\0\\0\\0\\0\\ \hline 0\\1\\ \hline 0\\0\\0\\0\\0\\0\\ \hline 0\\0\end{array}\right],\quad
(\boldsymbol{\alpha}_4^{\text{tr}})_3=\left[\begin{array}{c}0\\0\\0\\0\\0\\0\\ \hline 0\\0\\ \hline 0\\0\\0\\0\\0\\0\\ \hline 1\\0\end{array}\right],\quad
(\boldsymbol{\alpha}_4^{\text{tr}})_4=\left[\begin{array}{c}0\\0\\0\\0\\0\\0\\ \hline 0\\0\\ \hline 0\\0\\0\\0\\0\\0\\ \hline 0\\1\end{array}\right]
$$

Observe that the blocks of the matrices $\mathbb{A}_k^{\text{tr}}, \mathbb{B}_k^{\text{tr}}$, $k=1, 2, 3$, which have nonzero effect on the eigenvectors, $\boldsymbol{\alpha}_4^{\text{tr}}$ when the products $\mathbb{A}_k^{\text{tr}}\boldsymbol{\alpha}_4^{\text{tr}}$, $\mathbb{B}_k^{\text{tr}}\boldsymbol{\alpha}_4^{\text{tr}}$ are constructed, are the ones corresponding to spin-$\frac{1}{2}$ and reproduce the eigenvectors $(\alpha_4^{\text{tr}})_j$, $j=1, 2, 3, 4$, with a different order.

Example 2. Let us now consider the 20-dimensional, spin-$\frac{3}{2}$ equation with definite charge defined by the constants

$$
\begin{aligned}
&B=\frac{1}{2\sqrt{2}}, \qquad C=-\frac{1}{2\sqrt{2}}, \qquad Z=-\frac{1}{2\sqrt{2}}, \qquad K=\frac{1}{2\sqrt{2}}\\
&A=-\tfrac{1}{4}, \qquad \Gamma=\bar{\Gamma}=-\tfrac{1}{4}, \qquad \Theta=-\tfrac{1}{4}, \qquad \chi\neq 0
\end{aligned}
\tag{19}
$$

and based on the representation

$$
(\tfrac{1}{2},\tfrac{3}{2})\oplus(-\tfrac{1}{2},\tfrac{3}{2})\oplus(\tfrac{1}{2},\tfrac{5}{2})\oplus(-\tfrac{1}{2},\tfrac{5}{2})\oplus(\tfrac{1}{2},\tfrac{3}{2})\oplus(-\tfrac{1}{2},\tfrac{3}{2})
$$

proposed by Koutroules (1986). For this example again all the conditions for the existence of the secondary constraints are satisfied, as can be seen by using the explicit form of the matrices $\mathbb{L}_i$, $i=0, 1, 2, 3$, and the generators $\mathbb{A}_k$, $\mathbb{B}_k$, $k=1, 2, 3$, expressed for convenience in the spinor basis

$$\{\alpha_{11}^{\dot 1}, \alpha_{12}^{\dot 1}, \alpha_{22}^{\dot 1}, \alpha_{11}^{\dot 2}, \alpha_{12}^{\dot 2}, \alpha_{22}^{\dot 2}, d^{1}, d^{2}, \delta^{\dot 1}, \delta^{\dot 2},$$
$$b_1^{\dot 1 \dot 1}, b_1^{\dot 1 \dot 2}, b_1^{\dot 2 \dot 2}, b_2^{\dot 1 \dot 1}, b_2^{\dot 1 \dot 2}, b_2^{\dot 2 \dot 2}, c_1, c_2, \gamma_1, \gamma_2\} \tag{20}$$

There are eight subsidiary conditions of the second kind and this number is equal to the number of the spin-$\frac{1}{2}$ components $d^{\dot 1}$, $d^{\dot 2}$, $\delta^{\dot 1}$, $\delta^{\dot 2}$, c_1, c_2, γ_1, γ_2, which have to be constrained in order that the equation be a spin-$\frac{3}{2}$ wave equation. We notice that the number of subsidiary conditions of the second kind is equal to the number of linearly independent eigenvectors $\boldsymbol{\alpha}_4^{\text{tr}}$ corresponding to the zero eigenvalues of the matrices $(\mathbb{L}_i^{\text{tr}})^2$, $i=0$, 1, 2, 3, and such that conditions (17) are satisfied also. The matrices $(\mathbb{L}_i^{\text{tr}})^2$, $i=0$, 1, 2, 3, when considered in the spinor basis (20), have zero rows and columns corresponding to the spin-$\frac{1}{2}$ components and their eigenvectors $(\boldsymbol{\alpha}_4^{\text{tr}})_\omega$ corresponding to zero eigenvalues and satisfying (17) are

$$(\boldsymbol{\alpha}_4^{\text{tr}})_1 = \left[\begin{array}{c} 0\\0\\0\\0\\0\\0\\ \hline 1\\0\\ \hline 0\\0\\ \text{—}\\ 0\\0\\0\\0\\0\\0\\ \hline 0\\0\\ \hline 0\\0 \end{array}\right]\begin{array}{l} \\ \\ \\ \\ \\ \\ \text{spin-}\frac{1}{2}\\ \\ \text{spin-}\frac{1}{2}\\ \\ \\ \\ \\ \\ \\ \\ \\ \text{spin-}\frac{1}{2}\\ \\ \text{spin-}\frac{1}{2}\\ \\ \end{array}, \quad (\boldsymbol{\alpha}_4^{\text{tr}}) = \left[\begin{array}{c} 0\\0\\0\\0\\0\\0\\ \hline 0\\1\\ \hline 0\\0\\ \text{—}\\ 0\\0\\0\\0\\0\\0\\ \hline 0\\0\\ \hline 0\\0 \end{array}\right], \quad (\boldsymbol{\alpha}_4^{\text{tr}})_3 = \left[\begin{array}{c} 0\\0\\0\\0\\0\\0\\ \hline 0\\0\\ \hline 1\\0\\ \text{—}\\ 0\\0\\0\\0\\0\\0\\ \hline 0\\0\\ \hline 0\\0 \end{array}\right], \quad \boldsymbol{\alpha}_4^{\text{tr}})_4 = \left[\begin{array}{c} 0\\0\\0\\0\\0\\0\\ \hline 0\\0\\ \hline 0\\1\\ \text{—}\\ 0\\0\\0\\0\\0\\0\\ \hline 0\\0\\ \hline 0\\0 \end{array}\right]$$

$$(\boldsymbol{\alpha}_4^{\rm tr})_5=\left[\begin{array}{c}0\\0\\0\\0\\0\\0\\\hline 0\\0\\\hline 0\\0\\\hline 0\\0\\0\\0\\0\\0\\\hline 1\\0\\\hline 0\\0\end{array}\right],\quad(\boldsymbol{\alpha}_4^{\rm tr})_6=\left[\begin{array}{c}0\\0\\0\\0\\0\\0\\\hline 0\\0\\\hline 0\\0\\\hline 0\\0\\0\\0\\0\\0\\\hline 0\\1\\\hline 0\\0\end{array}\right],\quad(\boldsymbol{\alpha}_4^{\rm tr})_7=\left[\begin{array}{c}0\\0\\0\\0\\0\\0\\\hline 0\\0\\\hline 0\\0\\\hline 0\\0\\0\\0\\0\\0\\\hline 0\\0\\\hline 1\\0\end{array}\right],\quad(\boldsymbol{\alpha}_4^{\rm tr})_8=\left[\begin{array}{c}0\\0\\0\\0\\0\\0\\0\\0\\\hline 0\\0\\\hline 0\\0\\0\\0\\0\\0\\\hline 0\\0\\\hline 0\\1\end{array}\right]$$

Observe again that the blocks of the matrices $\mathbb{A}_k^{\rm tr}, \mathbb{B}_k^{\rm tr}$, $k=1, 2, 3$, that have nonzero effect on the eigenvectors $(\boldsymbol{\alpha}_4^{\rm tr})$ when the products $\mathbb{A}_k^{\rm tr}\alpha_4^{\rm tr}$, $\mathbb{B}_k^{\rm tr}\alpha_4^{\rm tr}$ are constructed are the ones corresponding to spin-$\frac{1}{2}$ and reproduce the vectors $(\boldsymbol{\alpha}_4^{\rm tr})$ with a different order.

REFERENCES

Gel'fand, I. M., Minlos, A. M., and Shapiro, Z. Ya. (1963). *Representations of the Rotation and the Lorentz Groups and Their Applications*, Pergamon Press, Oxford (1963).

Koutroulos, C. G. (1984). *Lettere al Nuovo Cimento*, **39**, 96.

Koutroulos, C. B. (1986a). *Naturforschung*, **41a**, 1024.

Koutroulos, C. G. (1986b). *Zeitschrift für Naturforschuns*, **41a**, 1029.

Koutroulos, C. G. (1986c). *International Journal of Theoretical Physics*, **25**, 1095.

Pauli, W., and Fierz, M. (1939). *Proceedings of the Royal Society*, **173A**, 211.

Velo, G., and Zwanziger, D. (1969). *Physical Review*, **186**, 1337.